

# A weakly universal cellular automaton in the heptagrid.

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## Abstract

In this paper, we construct a weakly universal cellular automaton in the heptagrid, the tessellation  $\{7, 3\}$  which is not rotation invariant but which is truly planar. This result, under these conditions, cannot be improved for the tessellations  $\{p, 3\}$ .

## 1 Introduction

This paper is basically an improvement of papers [9] and [10] where I proved the same result in the tessellations  $\{9, 3\}$  and  $\{8, 3\}$  respectively. The reason of this improvement lies in the relatively small number of rules for paper [9] and the fact, noticed in that paper, that several rules were uselessly duplicated. Also, as it is usual in this process of reducing the possibilities of the automaton, here its neighbourhood, it is needed to change something in the previous scenario of the simulation. Here, I repeat the scheme explained in [10]. The key idea of that paper was to combine two existing structures in order to eliminate one of the structures used so far in the simulation scheme explained in [3]. Note that the present result cannot be improved for this class of cellular automata which explicitly make use of non rotation invariant rules: indeed, the heptagrid is the tessellation  $\{p, 3\}$  of the hyperbolic plane with the smallest possible value for  $p$  which is 7. In this paper, I use the new system of coordinates introduced in [8] for the tilings  $\{p, 3\}$  and  $\{p-2, 4\}$ .

In this paper, the same model as in [6] and [5] and the other quoted papers is used.

In Section 2, I just indicate how the model is implemented in the heptagrid. In Section 3, I give the rules of the automaton, insisting in the way we defined these rules in a context where rotation invariance is no more required, which allows us to prove the following result:

**THEOREM 1** *There is a weakly universal cellular automaton on the heptagrid, the tessellation  $\{7, 3\}$  which is truly planar and which has two states.*

Presently, we turn to the proof of this result, repeating that the rules are not rotation invariant: the statement of the theorem does not mention that condition.

## 2 The scenario of the simulation

In the present section, I implement the standard structures used in [10]: tracks, the passive fixed switch, the fork, the doubler, the selector between a simple and a double locomotive, see further, the controller and the sensor.

I sketchily remember that we simulate a register machine by a railway circuit. Such circuit assembles infinitely many portions of straight lines, quarters of circles and switches. There are three kinds of switches, see [14, 3], for a description of the circuit and of its working. In [3] the simulation is thoroughly described and it is adapted to the hyperbolic context.

As in previous papers, the flip-flop and the memory switch are decomposed into simpler ingredients which we call sensors and control devices. This reinforces the importance of the tracks as their role for conveying key information is more and more decisive. Here too, tracks are blank cells marked by appropriate black cells we call **milestones**. We carefully study this point in Sub-section 2.1. Later, in Sub-section 2.2, we adapt the configurations described in [9] to the tessellation  $\{8, 3\}$ .

### 2.1 The tracks

In this implementation, the tracks are represented in a way which is a bit similar to that of [9, 10]. The present implementation is given by Figure 1.

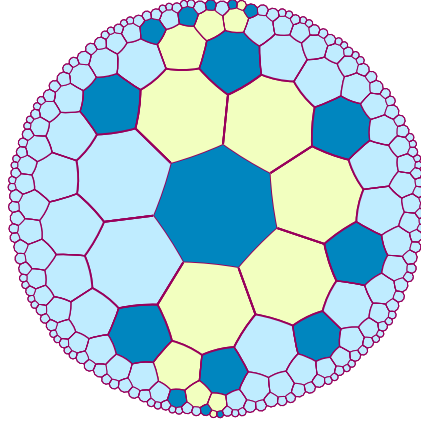
Here, we explicitly indicate the numbering of the sides in a cell which will be systematically used through the paper. As rotation invariance is no more required, we fix a side which will be, by definition, side 1 in the considered cell. This choice allows us to consider that the tracks are one way. The orientation is given by the side 1 of each cell which is, by convention, the side shared by the next cell on the track. All the other sides are numbered starting from this one and growing one by one while counter-clockwise turning around the cell. Note that, in our setting, the same side, which is shared by two cells, can receive two different numbers in the cells which share it. An example of this situation is given in Figure 1: in the central cell we denote by  $0(0)$ , side 1 is side 6 in the neighbour of the central cell sharing this side. In Sub-section 3.1, we go back to the construction of the tracks starting from the elements indicated in Figure 1. Note that Figure 1 shows us two rays starting from  $M$ , the mid-point of the side 2 of the central cell. These rays allow us to introduce the numbering of the tiles based on [8]. It will be used in the figures illustrating the paper.

The rays delimit what we call a **sector**. The rays are defined as follows. The ray  $u$  starts from the mid-point  $M$  of the side 2 of  $0(0)$  and it passes through the mid-point of its side 1. The ray also passes through the mid-point of the side 7 of the neighbour of  $0(0)$  which is seen through its side 1 and which we

In the figures of the paper, the central cell is the tile whose centre is the centre of the circle in which the figure is inscribed. The central cell is numbered by 0, denoted by 0(0). We number the sides of the tile as indicated in Figure 1. For  $i \in \{1..7\}$ , the cell which shares the side  $i$  with the central cell is called **neighbour**  $i$  and it is denoted by 1( $i$ ). The rotation around 0(0) allows us to attach a sector to each tile 1( $i$ ). Number 1 in this notation is the number given to the root of the tree attached to the sector defined from this tile, see [1, 8]. Here, that definition is adapted to the case of the tessellation  $\{7, 3\}$  which we call the heptagrid from now on as in many previous papers. We invite the reader to follow the present explanation on Figure 1. Consider the sector defined by the rays  $u$  and  $v$ . The neighbours of the cell 1(1) sharing its sides  $j$ ,  $j \in \{1..3\}$  are numbered  $j+1$  and are denoted by  $j+1(1)$ . We say that the cell 1(1) is a  $W$ -cell and its sons are defined by the rule  $W \rightarrow BWW$ , which means that 2(1) is a  $B$ -cell. This means that the sons of 2(1) are defined by the rule  $B \rightarrow BW$ , where the  $B$ -son has two consecutive sides crossed by  $u$  in their mid-points. These sons of 1(1) constitute the level 1 of the tree. The sons of 2(1), starting from its  $B$ -son are numbered 5 and 6 denoted by 5(1) and 6(1) respectively. By induction, the level  $n+1$  of the tree are the sons of the cells which lie on the level  $n$ , applying the above rules. The cells are numbered from the level 0, the root, level by level and, on each level from left to right, *i.e.* starting from the

ray  $u$  until the ray  $v$ . What we have seen on the numbering of the sons of  $2(1)$  is enough to see how the process operates on the cells. From now on, we use this numbering of the cells in the figures of the paper.

Further, Figure 2 illustrates how to assemble elements of the track on which the locomotive passes. As mentioned in the caption, the trajectory of the locomotive is illustrated in yellow. From the point of view of the cellular automaton, this is not a new state: yellow cells are blank cells. This representation is used to facilitate the understanding by the reader.



**Figure 2** *Element of the tracks: in yellow, the elements of the track where the locomotive passes.*

As can be seen in Figures of Section 3, the locomotive is implemented as a single black cell: it has the same colour as the milestones of the tracks. Only the position of the locomotive with respect to the milestones allows us to distinguish it from the milestones. As clear from the next sub-section, we know that besides this **simple locomotive**, the locomotive also occurs as a **double one** in some portions of the circuit: two consecutive black cells. In a double locomotive, we call its first, second cell the **front**, **rear** respectively of the locomotive. In a simple locomotive, front and rear are the same cell which will be called front in that case. We reserve the word rear for the second cell of a double locomotive.

We can see that in that figure that, assuming that the locomotive goes from top to bottom, the milestones are in neighbours 2, 5 and 7 as for the cell 1(6) or in neighbours 2, 4 and 6 as for the cell 3(1). It is important to notice that such tracks allow us to join any pair of points. In Section 3, we shall check that the rules will satisfy this constraint.

The circuit also makes use of signals which are implemented in the form of a simple locomotive. So that at some point, it may happen that we have three simple locomotives travelling on the circuit: the locomotive and two auxiliary signals involved in the working of some switch. For aesthetic reasons, the black



colour which is opposed to the blank is dark blue in the figures.

## 2.2 The structures of the simulation

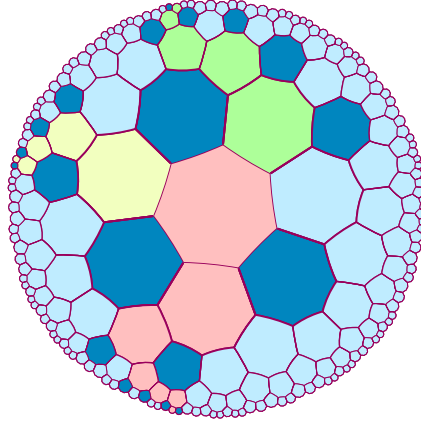
The crossings of [14] are present in many ones of my papers. Starting from [2], I replaced the crossing by roundabouts, a road traffic structure, in my simulations in the hyperbolic plane. At a roundabout where two roads are crossing, if you want to keep the direction arriving at the roundabout, you need to leave the roundabout at the second road. I refer the reader to [3] for references. The structure is a complex one, which requires a fixed switch, a doubler and a selector. Other structures are used to simulate the switches used in [14, 3]: the fork, the controller and the sensor. In this section, we present the implementation of these structures which are those of [10] adapted to the present tessellation.

### 2.2.1 The fixed switch, the doubler and the fork

We look at the fixed switch first, and then at the doubler and the fork as the doubler is a combination of the fork and of the fixed switch.

#### The fixed switch

As the tracks are one-way and as an active fixed switch always sends the locomotive in the same direction, no track is needed for the other direction: there is no active fixed switch. Now, passive fixed switches are still needed as just seen in the previous paragraph.



**Figure 3** *Idle configuration of the passive fixed switch. In yellow, the arriving path from the left, in pink, the arriving path from the right, in pink, the path leaving the switch.*

Figure 3 illustrates the passive fixed switch when there is no locomotive around: we say that such a configuration is **idle**. We shall again use this term

in the similar situation for the other structures and for individual cells too.

We can see that it consists of elements of the tracks which are simply assembled in the appropriate way in order to drive the locomotive to the bottom direction in the picture, whatever upper side the locomotive arrived at the switch. The path followed by the locomotive to the switch is in yellow or in green until the central cell which is pink. The path from the left-hand side, yellow in the figure, consists, in this order of the cells 29(2), 11(2), 10(2), 3(2) and 1(2). From the right-hand side, green in the figure, it consists of the cells 23(1), 9(1), 3(1), 2(1) and 1(7). Of course, 1(2) and 1(7) are neighbours of 0(0). The path followed by the locomotive from 0(0) is in pink in the figure. It consists of the following cells in this order: 0(0), 1(4), 2(4), 7(4), 8(4), 9(4) and 24(4). Note that the cell 0(0) in Figure 3 is a standard element of the track with three milestones in 1(5), 1(1) and 1(3), its neighbours 2, 5 and 7 respectively. Note that 1(3) and 1(1) are milestones for the cell 1(2), that 1(3) and 1(5) are milestones for 1(4) and that 2(7) and 1(1) are milestones for 1(7). Note that the milestones of 1(7) are its neighbours 3, 5 and 7.

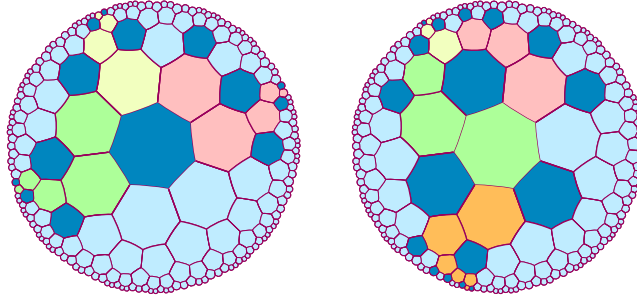
From our description of the working of the round-about, a passive fixed switch must be crossed by a double locomotive as well as a simple one. Later, in Subsection 3.2.1, we shall check that the structure illustrated by Figure 3 allows those crossings.

### The doubler and the fork

The fork is the structure illustrated by the left-hand side picture of Figure 4. Note that its structure is very different from that of the tracks or of the fixed switch. The central cell 0(0) is black and two paths start from 1(1), each one on one side of the central cell with respect to its axis which crosses its side 1 and passes through the vertex which is opposite to side 1, it is shared by sides 4 and 5. The paths take each one two cells around 0(0) and then leave the neighbourhood of the cell 0(0). The cells leading to 1(1) are yellow in the figure. The left-hand side track is green, consisting of the following cells, in this order: 1(2), 1(3), 3(3), 7(3) and 20(3). The right-hand side track is pink. It consists of the cells: 1(7), 1(6), 4(6), 5(7) and 13(7). The locomotive, a simple one, arrives through the yellow path: 28(1), 10(1), 4(1), and 1(1). From 1(1), two simple locomotives appear: one in 1(2), going on along the green path, the other in 1(7), travelling along the pink path.

The doubler is a structure which receives a simple locomotive and which yields a double one. The idea is to use a fork to produce two simple locomotives and then to gather them at a fixed switch in order to produce the double locomotive. The process is illustrated by the right-hand side of Figure 4. The structure is inspired by that of [10] but it turns out that here, it is much simpler than there. The reason is that in [10], the even number of sides compelled me to devise a detour in order that two locomotives arrive at the same time one after the other at one entrance of the fixed switch. In the heptagrid, the odd number of sides allowed me to perform a simpler implementation. The odd number allows us to have two equal paths around the common milestones of the

concerned elements of the tracks. It is enough to place the central cell of the fixed switch at the end of one of the paths, the green path on the right-hand side picture of Figure 4. The picture uses the same colours as the picture of the fork with the same meaning. Consider the green path. Its cells are, in this order: 2(2), 1(2) and 0(0) which makes three cells. The pink path consists of the following cells, in this order: 3(1), 2(1) and 1(7). We can see that the cells around 0(0) are exactly the neighbours of the central cell of a fixed switch, see Figure 3. According to this description, the two simple locomotives created at the same time in 2(2) and 3(1) respectively do not arrive at the same time at the cell 0(0). When the locomotive created in 2(2) arrives at the cell 0(0), the locomotive created in 3(1) is at 1(7), so that the two black cells in 1(7) and 0(0) constitute a double locomotive arriving from the right whose front is in the central cell of the fixed switch. Then the double locomotive leaves the switch through the orange path : in this order, 1(4), 2(4), 7(4), 8(4), 9(4) and 24(4). Accordingly the structure works as expected for a doubler. Note that elements of the track only are involved.

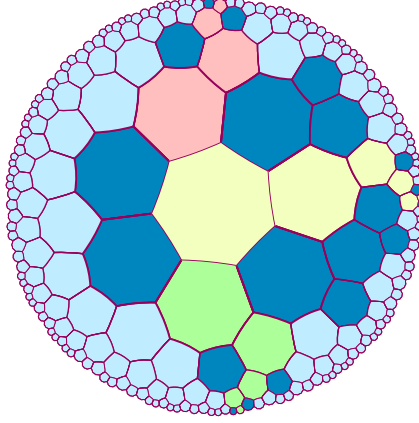


**Figure 4** *Idle configurations. To left: the fork. To right: the doubler. In both of them: arrival of the locomotive through the yellow track. Then, one locomotive on the green and on the pink tracks. In the doubler: the double locomotive leaves the switch through the orange track.*

### 2.2.2 The selector

The selector is illustrated by Figure 5. This structure is less symmetric than the corresponding structure of [10], which makes another difference with that paper. We have a yellow track through which the locomotive arrives, simple or double, both cases are possible. When a simple locomotive arrives, it leaves the cell through 1(1), via the pink path which consists of the cells 1(1), 2(1), 7(1) and 18(1). When a double locomotive arrives, a simple locomotive leaves the structure through the green path, the cells: 1(4), 2(5), 5(5), 12(4) and 33(4). Both cells 1(5) and 1(7) can detect whether the locomotive is simple or double. They can do that when the front of the locomotive is in 0(0). Then, if the locomotive is double, its rear is in 1(6). Both cells 0(0) and 1(6) are neighbours of 1(5) and of 1(7) too.

In Subsubsection 2.2.2, the rules will show that such a working will be observed.



**Figure 5** *Idle configuration of the selector. The cells 1(7) and 1(5) detect whether the locomotive is simple or double. Arriving through the yellow track, a simple locomotive leaves through the pink track, a double locomotive leaves through the green track as a simple locomotive.*

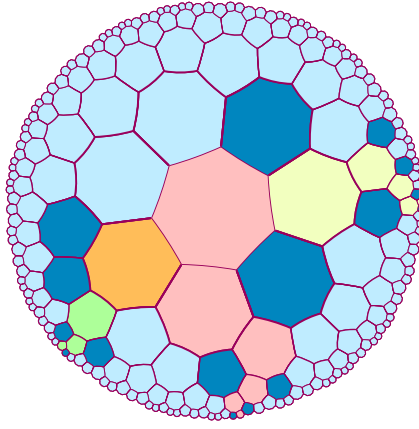
## 2.3 The controller and the sensor

In this Sub-section, we look at the additional structures used for the flip-flop and for the memory switch, see [14, 3] for the definitions and for the implementation in the hyperbolic plane. As explained in [3], the flip-flop and the active memory switch are implemented by using the fixed switch, the fork and a new structure we shall study in Subsubsection 2.3.1: the **controller**. The structure is illustrated by Figure 6. For the passive memory switch, we need the fork, the fixed switch and another new structure we shall study in Subsubsection 2.3.2: the **sensor** illustrated by Figure 7.

### 2.3.1 The controller

As shown by Figure 6, the controller sits on an ordinary cell of the track. The locomotive which runs on that track is always simple. The track consists of the yellow path which passes through 25(6), 9(6), 10(6), 4(6) and 1(6), a neighbour of 0(0), and of the pink path which starts from 0(0) and which is crossed by the locomotive when the controller is black. The **colour** of the controller is defined by the cell 1(3), in orange in Figure 6. The pink path consists of the following cells, in this order: 0(0), 1(4), 2(5), 5(5), 12(4) and 33(4), a path already seen in previous figures. When the cell 1(3) is black, then the cell 0(0) is an ordinary element of the track, so that the locomotive goes on its way along the pink path, leaving the controller. If the cell 1(3) is white, then the cell 0(0) can no more work as an element of the track. It remains white, which means

that the locomotive is stopped at 1(6): after that, it vanishes. This corresponds to the working of a selection in an active passage of the switch: the locomotive cannot run along a non-selected track. Here it can do it for a while, but at some point, it is stopped by the controller. Note that the occurrence of a locomotive in the structure does not change the colour in 1(3). The change of colour in that cell is performed by a signal which takes the view of a simple locomotive arriving through another track: 31(3), 12(3) and 4(3), that latter cell being a neighbour of 1(3). When the locomotive-signal arrives at 4(3), it makes the cell 1(3) change its colour: from white to black and from black to white.



**Figure 6** *Idle configuration of the controller of the flip-flop and of the active memory switch. In orange, the cell 1(3): the sensor which controls the working of the device. In pink, the portion of the track which is allowed when the cell 1(3) is black only.*

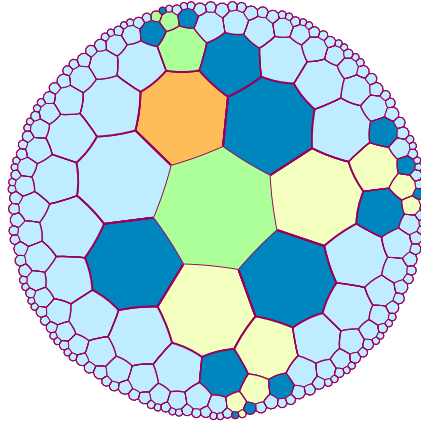
### 2.3.2 The sensor

Let us now turn to the sensor, illustrated by Figure 7. As suggested by its name, the sensor does not exactly behave like the controller. When the locomotive passes on the non-selected track, it is not stopped. A fork creates two simple locomotives, see [9, 10]: one of them goes on to the switch, the other to the sensor which uses it as a messenger for the signal it has to send to the active switch in order to change the selection of the tracks.

This is illustrated by the structure of the figure. The path is the same as in Figure 6. The cell which plays the role of a sensor is this time the cell 1(1) whose state we call the **colour** of the sensor. Note that the neighbourhood of that cell in Figure 7 is the same, up to rotation, to the neighbourhood of the cell 1(3) in Figure 6: the green path here consists of the cells 26(1), 9(1) and 3(1) the latter being a neighbour of 1(1).

Figure 7 shows a very different structure for the cell 0(0) compared with

that of Figure 6. When the sensor is white, its neighbourhood is exactly that of the cell  $0(0)$  when the controller is black: it is an ordinary element of the track so that the locomotive goes on its way on the track. The difference in both structures lies in the logic of the switches. In the case of the controller, when the locomotive goes on its way, it is the locomotive of the circuit going to another switch or to a round-about. In the case of the sensor, the locomotive which goes on its way on the track becomes a signal sent to the active switch associated to the passive switch.



**Figure 7** *Idle configuration of the sensor of the passive memory switch. In yellow, the cell  $1(1)$ , the sensor-cell.*

We can just note that the change of colour is different in the sensor: when the sensor is white, if a locomotive passes, it must become black: the signal is the locomotive itself, as will be seen in the Subsection 3.5. This is why the cell  $0(0)$  is green in Figure 7: the cell  $1(1)$  can see the locomotive only when it is in  $0(0)$ . When the sensor is black, it has to be changed if a locomotive passed through the other sensor which then changed from white to black. The locomotive which arrived at the formerly white sensor is sent to the still black one in order to make it change to white. The locomotive arrives through the green path of Figure 7. As the configuration is the same around the cell  $1(1)$  of that Figure as that around  $1(3)$  in Figure 6, the change from black to white is performed.

### 3 Rules

The figures of Section 2 help us to establish the rules. Their application is illustrated by figures of this section which were drawn by a computer program which checked the coherence of the rules. The program also wrote the PostScript files of the pictures from the computation of the application of the rules to the configurations of the various types of parts of the circuit. The computer program

also established the traces of execution which contribute to the checking of the application of the rules.

Let us explain the format of the rules and what is allowed by the relaxation from rotation invariance. We remind the reader that a rule has the form  $\underline{x}_o x_1 \dots x_7 \underline{x}_n$ , where  $x_o$  is the state of the cell  $c$ ,  $x_i$  is the **current** state of the neighbour  $i$  of  $c$  and  $x_n$  is the **new** state of  $c$  applied by the rule. As the rules no more observe the rotation invariance, we may freely choose which is side 1 for each cell. We take this freedom from the format of a rule which only requires to know which is neighbour 1. In order to restrict the number of rules, it is decided that, as a general rule, for a cell which is an element of the track, side 1 is the side shared by the cell and its next neighbour on the track, so that tracks are one way, as already mentioned. There can be exceptions when the cell is in a switch or the neighbour of the central cell in a switch. In particular, when a cell belongs to two tracks, side 1 is arbitrarily chosen among the two possible cases. At some places, side 1 may be chosen in order to allow the cellular automaton to apply the expected rule. The milestones have their side 1 shared with an element of the track which also contributes to reduce the number of rules.

We have to keep in mind that there are two types of rules. Those which keep the structure invariant when it is idle, we call this type of rules **conservative**, and those which control the motion of the locomotive. Those latter rules, which we call **motion rules**, are the rules applied to the cells of the tracks as well as their milestones and, sometimes to the cells of the structures which may be affected by the passage of the locomotive. Next, in each sub-section, we give the rules for the motion of the locomotive in the tracks, then for the fixed switch, then for the doubler and for the fork, then for the selector, then for the controller and, eventually, for the sensor. In each sub-section, we also illustrate the motion of the locomotive in the structure as well as a table giving traces of execution for the cells of the track involved in the crossing.

### 3.1 The rules for the tracks

Figure 1 shows us a single element of the track. Figure 2 shows us how to assemble elements as illustrated in Figure 1 in order to constitute tracks. In Figure 2, see page 4. In that figure, the tracks is represented by the yellow cells, in this order when going from top to bottom: the cells 16(1), 6(1), 7(1), 3(1), 1(1), 1(7), 1(6), 1(5), 1(4), 3(4), 10(4), 11(4) and 29(4).

A close look at the tracks shows us at least two kinds of cells despite all of them are three-milestoned cells, another difference with [10] where four-milestoned elements of the tracks are often present. In Figure 2, there are three-milestoned cells with milestones in their neighbours 2, 4 and 7 as, for instance, 1(6) and 3(1), three-milestoned cells with their milestones in neighbours 3, 5 and 7, as 1(1) and 7(1) for instance. Table 4 shows us that for the cells 1(6) and 3(1), rules 4, 36, 17, 25 are applied, while rules 3, 38, 40, 43 are applied for the cells 1(1) and 7(1). Here and later, we write in red the number of a rule whose new state is different from the current state of the rule. Another assortment of the milestones is in neighbours 2, 5 and 7. Table 1 gives the

motion rules corresponding to these cells and from a few others as indicated by Tables 4 and 5. The rules are borrowed from Table 2 which displays all rules used by the locomotives on the tracks.

Note that Table 1 gives all possible neighbourhoods for three isolated milestones, requiring that neighbour 1 be blank in the idle configuration. For instance, take the positions 2, 4, 6 for the milestones. Rule 14 is the conservative rule and rule 31 corresponds to the case of a simple locomotive being in the cell. Now, as the locomotive always leaves the cell through neighbour 1, rule 34 applies after rule 31. Now, there are *a priori* three possible entrances for the locomotive : neighbour 5 as in rule 28 displayed in Table 1. Table 2 also shows that neighbour 3 and 7 are also used, look at rules 63, WWBBWBWB and 41, WWBWBWB. The same observation can be made for the other dispatches of the milestones in Table 1. In this table, we also mark the position of the locomotive as B.

**Table 1** *The motion rules for a simple locomotive.*

2, 4, 6		2, 4, 7		2, 5, 7		3, 5, 7	
14	<u>WWBWBWBW</u>	4	<u>WWBWBWBW</u>	7	<u>WWBWBWBW</u>	3	<u>WWBWBWBW</u>
28	<u>WWBWBWBW</u>	36	<u>WWBWBWBW</u>	16	<u>WWBWBWBW</u>	38	<u>WWBWBWBW</u>
31	<u>BWBWBWBW</u>	17	<u>BWBWBWBW</u>	24	<u>BWBWBWBW</u>	40	<u>BWBWBWBW</u>
34	<u>BWBWBWBW</u>	25	<u>BWBWBWBW</u>	29	<u>BWBWBWBW</u>	43	<u>BWBWBWBW</u>

**Table 2** *Rules managing the motion of a locomotive on the tracks.*

from down to top : simple locomotive							
1	<u>WWWWWWW</u>	13	<u>WBWWWWW</u>	25	<u>WBWBWBWB</u>	37	<u>BWWBWBWB</u>
2	<u>BWWWWWW</u>	14	<u>WBWBWBWB</u>	26	<u>BWBWBWBW</u>	38	<u>WWBWBWBW</u>
3	<u>WWBWBWBW</u>	15	<u>BWBWBWBW</u>	27	<u>WWBWBWBW</u>	39	<u>BWBWBWBW</u>
4	<u>WBWBWBWB</u>	16	<u>WBWBWBWB</u>	28	<u>WBWBWBWB</u>	40	<u>BWBWBWBW</u>
5	<u>WWBWBWBW</u>	17	<u>BWBWBWBW</u>	29	<u>WBWBWBWB</u>	41	<u>WWBWBWBW</u>
6	<u>BWBWBWBW</u>	18	<u>WBWBWBWB</u>	30	<u>WBWBWBWB</u>	42	<u>WBWBWBWB</u>
7	<u>WBWBWBWB</u>	19	<u>BWBWBWBW</u>	31	<u>BWBWBWBW</u>	43	<u>WBWBWBWB</u>
8	<u>WBWBWBWB</u>	20	<u>WBWBWBWB</u>	32	<u>WBWBWBWB</u>	44	<u>BWBWBWBW</u>
9	<u>WBWBWBWB</u>	21	<u>WBWBWBWB</u>	33	<u>WBWBWBWB</u>	45	<u>WBWBWBWB</u>
10	<u>WWWWWWW</u>	22	<u>BWBWBWBW</u>	34	<u>WBWBWBWB</u>		
11	<u>WWBWBWBW</u>	23	<u>BWBWBWBW</u>	35	<u>WBWBWBWB</u>		
12	<u>BWBWBWBW</u>	24	<u>BWBWBWBW</u>	36	<u>WBWBWBWB</u>		
from down to top: double locomotive							
46	<u>BWBWBWBW</u>	51	<u>BBWBWBWB</u>	56	<u>BWBWBWBW</u>	60	<u>BWBWBWBW</u>
47	<u>BWBWBWBW</u>	52	<u>BWBWBWBW</u>	57	<u>BWBWBWBW</u>	61	<u>BWBWBWBW</u>
48	<u>BBWBWBWB</u>	53	<u>BBWBWBWB</u>	58	<u>BWBWBWBW</u>	62	<u>BWBWBWBW</u>
49	<u>BBWBWBWB</u>	54	<u>BWBWBWBW</u>	59	<u>BWBWBWBW</u>		
50	<u>BWBWBWBW</u>	55	<u>WBWBWBWB</u>				
from top to bottom, when the locomotive is:							
simple				double			
63	<u>WBWBWBWB</u>	64	<u>BWBWBWBW</u>	65	<u>BWBWBWBW</u>	66	<u>BWBWBWBW</u>

Accordingly, for the display 2, 4, 7, besides rule 36 corresponding to an



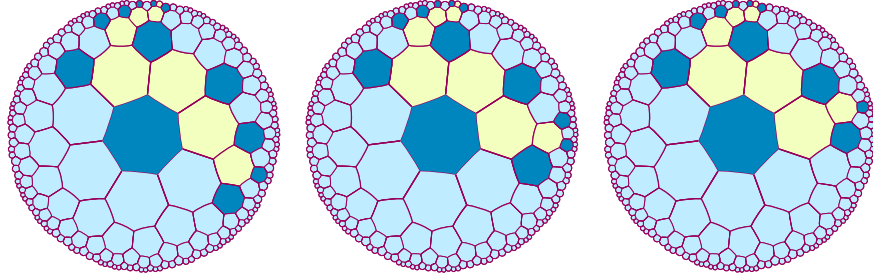
entrance through neighbour 3, rule 41 again corresponds to another one through neighbour 6, and rule 45  $\underline{W}BWB\overline{B}WB\overline{B}$  corresponds to again another one through neighbour 5. This can be repeated for the other neighbours, see Table 3 where the number in brackets indicates the neighbour through which the locomotive enters. In that table the front of the locomotive is marked in the rules as  $\underline{B}$ . Rule 71 will be used later, in the fixed switch.

**Table 3** *The other rules involved for the motion of a simple locomotive.*

3, 5, 7 :	38	$\underline{W}WBWB\overline{B}B\overline{B}$	[6]	71	$\underline{W}WB\overline{B}BWB\overline{B}$	[4]	32	$\underline{W}WB\overline{B}WBWB\overline{B}$	[2]
2, 5, 7 :	16	$\underline{W}WBWB\overline{B}B\overline{B}$	[6]	45	$\underline{W}WB\overline{B}BWB\overline{B}$	[4]	32	$\underline{W}WB\overline{B}WBWB\overline{B}$	[3]
2, 4, 7 :	36	$\underline{W}WB\overline{B}BWB\overline{B}$	[3]	41	$\underline{W}WBWB\overline{B}B\overline{B}$	[6]	45	$\underline{W}WBWB\overline{B}WB\overline{B}$	[5]
2, 4, 6 :	28	$\underline{W}WBWB\overline{B}BWB\overline{B}$	[5]	63	$\underline{W}WB\overline{B}BWB\overline{B}$	[3]	41	$\underline{W}WBWB\overline{B}B\overline{B}$	[7]

Call the rules of Table 1 for a given neighbourhood, the conservative rule, the front rule, the cell rule and the witness rule, the names being self explanatory.

Table 1 also shows us an interesting features: the neighbourhood of rule 36 is  $\overline{W}B\overline{B}BWB$  and that of rule 29 is  $B\overline{B}WBWB$ . It is not difficult to see that we can pass from one neighbourhood to the other by a circular permutations. Accordingly, rule 36 and rule 29 are not rotationally compatible: rule 36 is a front rule, rule 29 is a witness one. The other conclusion we can draw from the comments regarding other variants of the rule allowing to make the locomotive enter the cell is that this number of variants gives us an important flexibility for devising tracks which go from a tile to another one. On the example of Figure 2 we can see that the track going to 1(6) can also go to 2(6), 3(6) or 4(6). Figure 8 show us how to proceed to continue a path to the sons of an already reached element.



**Figure 8** *Element of the tracks: in yellow, the elements of the track where the locomotive passes.*

In Figure 2, the neighbourhood of 1(6) is of the type 2, 5, 7. On the pictures of Figure 8, we have the following neighbourhoods:

1(6):	2, 4, 6	3, 5, 7	2, 4, 7
sons):	2(6)	3(6)	4(6)
	2, 4, 6	2, 5, 7	2, 5, 7

It is worth noticing that many rules appearing in Table 4 showing the rules used when a simple locomotive goes up along the tracks illustrated by Figure 2 also appear in Table 5 which display the rules used when the same locomotive goes down, assuming that the sides 1 have been changed in order to allow the motion from top to bottom on the same cells. Of course, in the circuit, for any cell, side 1 is fixed once and for all.

**Table 4** *Execution of the rules 1 up to 45: motion along the tracks from down to top for a simple locomotive.*

	11 <sub>4</sub>	10 <sub>4</sub>	3 <sub>4</sub>	1 <sub>4</sub>	1 <sub>5</sub>	1 <sub>6</sub>	1 <sub>7</sub>	1 <sub>1</sub>	3 <sub>1</sub>	7 <sub>1</sub>	6 <sub>1</sub>
1	25	24	16	14	7	4	4	3	4	3	7
2	4	29	24	28	7	4	4	3	4	3	7
3	4	7	29	31	32	4	4	3	4	3	7
4	4	7	7	34	24	36	4	3	4	3	7
5	4	7	7	14	29	17	36	3	4	3	7
6	4	7	7	14	7	25	17	38	4	3	7
7	4	7	7	14	7	4	25	40	41	3	7
8	4	7	7	14	7	4	4	43	17	38	7
9	4	7	7	14	7	4	4	3	25	40	45

**Table 5** *Execution of the rules 1 up to 45: motion along the tracks from top to bottom for a simple locomotive.*

	6 <sub>1</sub>	7 <sub>1</sub>	3 <sub>1</sub>	1 <sub>1</sub>	1 <sub>7</sub>	1 <sub>6</sub>	1 <sub>5</sub>	1 <sub>4</sub>	3 <sub>4</sub>	10 <sub>4</sub>	11 <sub>4</sub>
1	34	24	63	7	7	7	3	7	4	4	7
2	14	29	31	32	7	7	3	7	4	4	7
3	14	7	34	24	16	7	3	7	4	4	7
4	14	7	14	29	24	16	3	7	4	4	7
5	14	7	14	7	29	24	38	7	4	4	7
6	14	7	14	7	7	29	40	45	4	4	7
7	14	7	14	7	7	7	43	24	36	4	7
8	14	7	14	7	7	7	3	29	17	36	7

Before turning to the rules for a double locomotive, note that in the conservative rules for the elements of the tracks the neighbourhoods are rotated forms of each other.

The rules for the double locomotive are displayed in Table 2. They can be derived from the previous rules as follows. The conservative and the front rules

are the same: at those times, the cell do not know whether the locomotive is simple or double. Once the front is in the cell, the cell rule cannot be applied as the rear is seen at the place were the front was by one time backwards. Accordingly the cell rule is replaced by two new rules: the rear rule, which makes the second cell of the locomotive enter the cell and the clearing rule which makes it leave the cell. Note that the rear, clearing rules are obtained from the front, witness rules respectively by changing the current state from  $w$  to  $B$ . Table 6 gives the rules applied in the respective neighbourhoods considered in Table 1. That table has the same property as Table 1. Other entries are possible for the double locomotive. Table 3 indicates us for each neighbourhood the sequence of rules constituted, in this order, by the front, the rear and the clearing rules. Also, brackets indicate, after each triple, which is the neighbour through which the locomotive enters the cell. In that table, the locomotive is marked as  $B$ . In Table 6, the front of the locomotive is  $B$  and its rear is  $B$ . Table 7 indicates all possible neighbourhoodes with, in brackets the entry of the front of the locomotive. Note the same phenomenon as in Table 1: a few pair of rear and clearing rules are the same for different neighbours. In fact that feature appears when two neighbours differ by one place, for example 2, 4, 7 and 2, 5, 7. Clearly, the positions of the black neighbours is the same for the front and rear rules when the entrance is neighbour 5 and 4 respectively.

**Table 6** *The motion rules for a double locomotive.*

2, 4, 6		2, 4, 7		2, 5, 7		3, 5, 7	
14	$\underline{w}wBwBwBw$	4	$\underline{w}wBwBwBw$	7	$\underline{w}wBwBwBw$	3	$\underline{w}wBwBwBw$
28	$\underline{w}wBwBwBw$	36	$\underline{w}wBwBwBw$	16	$\underline{w}wBwBwBw$	38	$\underline{w}wBwBwBw$
52	$\underline{w}wBwBwBw$	56	$\underline{w}wBwBwBw$	47	$\underline{w}wBwBwBw$	59	$\underline{w}wBwBwBw$
53	$\underline{w}wBwBwBw$	48	$\underline{w}wBwBwBw$	51	$\underline{w}wBwBwBw$	60	$\underline{w}wBwBwBw$
34	$\underline{w}wBwBwBw$	25	$\underline{w}wBwBwBw$	29	$\underline{w}wBwBwBw$	43	$\underline{w}wBwBwBw$

**Table 7** *The other rules involved for the motion of a double locomotive.*

3, 5, 7 : 38, 59, 60 [6]	71, 72, 60 [4]	32, 54, 60 [2]
2, 5, 7 : 16, 47, 51 [6]	45, 66, 51 [4]	32, 53, 51 [3]
2, 4, 7 : 36, 56, 48 [3]	41, 61, 48 [6]	45, 67, 48 [5]
2, 4, 6 : 28, 52, 53 [5]	63, 65, 53 [3]	41, 61, 53 [7]

Table 6 contains 20 rules. The cell rule is different in Table 1 so that, together, those tables contain 24 rules. Table 7 brings in 16 new rules. Accordingly we have 40 rules for the elements of the track only. There are other rules concerning the milestones:  $\underline{B}w^\alpha Bw^\beta B$  with  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 6$  together with  $\alpha < 5$  when a simple locomotive moves, see rules 19, 22, 15, 37 and 39. For a double one, we have all possible rules of the form  $\underline{B}w^\alpha Bw^\beta B$  with  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 5$  together with  $\underline{B}Bw^\beta B$ , see rules 49, 46, 57, 58, 64, 62 and 50. We have also to

append rule 1,  $\underline{ww}^7\underline{w}$ , the conservative rule of the blank cells which have no black cell among their neighbours, as well as rule 2,  $\underline{bw}^7\underline{b}$ , which is the conservative rule of the milestones of the elements of the track. Table 8 illustrates the use of many of those rules together with some others for four cells: 4(1), a white cell, and three milestones: 2(1), 0(0) and 4(4). The cell 4(1) illustrates a situation when a white cell can see two consecutive elements of the tracks. We use the same colour conventions as in Tables 1 and 6 for the front and for the rear of a locomotive.

It can be noted that the rules of Table 8 do not change the current state of the cell: it is conformal with the role of witness devoted to these cells. We also can remark that the change of direction in the motion boils down to change the order of application of the rules. We also can see the change in the display of the colours in the rules attached to the motion of a double locomotive.

**Table 8** Rules for cells witnessing the motion on the tracks.

4(1):	simple $\uparrow$	double $\uparrow$	simple $\downarrow$	double $\downarrow$
	5 $\underline{wwwBwwwBw}$	5 $\underline{wwwBwwwBw}$	5 $\underline{wwwBwwwBw}$	5 $\underline{wwwBwwwBw}$
	35 $\underline{wBwBwwwBw}$	35 $\underline{wBwBwwwBw}$	33 $\underline{wBwBwwwBw}$	33 $\underline{wBwBwwwBw}$
	33 $\underline{wBwBwwwBw}$	55 $\underline{wBwBwwwBw}$	35 $\underline{wBwBwwwBw}$	55 $\underline{wBwBwwwBw}$
	5 $\underline{wwwBwwwBw}$	33 $\underline{wwwBwwwBw}$	5 $\underline{wwwBwwwBw}$	35 $\underline{wBwBwwwBw}$
		5 $\underline{wwwBwwwBw}$		5 $\underline{wwwBwwwBw}$
2(1):	simple $\uparrow$	double $\uparrow$	simple $\downarrow$	double $\downarrow$
	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$
	22 $\underline{BwBwwwwwwB}$	22 $\underline{BwBwwwwwwB}$	39 $\underline{BwwwwwwBWB}$	39 $\underline{BwwwwwwBWB}$
	19 $\underline{BwwwwwwWB}$	49 $\underline{BwwwwwwWB}$	44 $\underline{BwwwwwwWB}$	64 $\underline{BwwwwwwBWB}$
	23 $\underline{BwwwwwwBB}$	50 $\underline{BwwwwwwBB}$	23 $\underline{BwwwwwwBB}$	62 $\underline{BwwwwwwBB}$
	44 $\underline{BwwwwwwBWB}$	62 $\underline{BwwwwwwBBB}$	19 $\underline{BwwwwwwWB}$	50 $\underline{BwwwwwwBBB}$
	39 $\underline{BwwwwwwWB}$	64 $\underline{BwwwwwwBWB}$	22 $\underline{BwBwwwwwwB}$	49 $\underline{BBBwwwwwwB}$
	2 $\underline{BwwwwwwWB}$	39 $\underline{BwwwwwwBWB}$	2 $\underline{BwwwwwwWB}$	22 $\underline{BwBwwwwwwB}$
		2 $\underline{BwwwwwwWB}$		2 $\underline{BwwwwwwWB}$
0(0):	simple $\uparrow$	double $\uparrow$	simple $\downarrow$	double $\downarrow$
	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$
	19 $\underline{BwwwwwwWB}$	19 $\underline{BwwwwwwWB}$	39 $\underline{BwwwwwwBWB}$	39 $\underline{BwwwwwwBWB}$
	22 $\underline{BwBwwwwwwB}$	49 $\underline{BwwwwwwWB}$	37 $\underline{BwwwwwwWB}$	58 $\underline{BwwwwwwBWB}$
	15 $\underline{BwBwwwwwwB}$	46 $\underline{BwBwwwwwwB}$	15 $\underline{BwBwwwwwwB}$	57 $\underline{BwwwwwwBWB}$
	37 $\underline{BwwwwwwWB}$	57 $\underline{BwwwwwwBWB}$	22 $\underline{BwBwwwwwwB}$	46 $\underline{BwBwwwwwwB}$
	39 $\underline{BwwwwwwBWB}$	58 $\underline{BwwwwwwBWB}$	19 $\underline{BwwwwwwWB}$	49 $\underline{BBBwwwwwwB}$
	2 $\underline{BwwwwwwWB}$	39 $\underline{BwwwwwwBWB}$	2 $\underline{BwwwwwwWB}$	19 $\underline{BBBwwwwwwB}$
		2 $\underline{BwwwwwwWB}$		2 $\underline{BwwwwwwWB}$
4(4):	simple $\uparrow$	double $\uparrow$	simple $\downarrow$	double $\downarrow$
	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$	2 $\underline{BwwwwwwWB}$
	15 $\underline{BwBwwwwwwB}$	15 $\underline{BwwwwwwWB}$	23 $\underline{BwwwwwwBB}$	23 $\underline{BwwwwwwBB}$
	22 $\underline{BwBwwwwwwB}$	46 $\underline{BwBwwwwwwB}$	19 $\underline{BwwwwwwWB}$	50 $\underline{BBwwwwwwBB}$
	19 $\underline{BwwwwwwWB}$	49 $\underline{BBBwwwwwwB}$	22 $\underline{BwBwwwwwwB}$	49 $\underline{BBBwwwwwwB}$
	23 $\underline{BwwwwwwBB}$	50 $\underline{BBwwwwwwBB}$	15 $\underline{BwBwwwwwwB}$	46 $\underline{BwBwwwwwwB}$
	2 $\underline{BwwwwwwWB}$	23 $\underline{BwwwwwwBB}$	2 $\underline{BwwwwwwWB}$	15 $\underline{BwwwwwwWB}$
		2 $\underline{BwwwwwwWB}$		2 $\underline{BwwwwwwWB}$

Tables 9 and 10 show which instructions are applied to the cells of the track when a double locomotive passes: from down to top in Table 9, from top to bottom in Table 10. Figure 9 illustrates the motion of both types of locomotives when they go from down to top and Figure 10 does the same when they go from

top to bottom.

We conclude this section by a remark: in [9] and in [10], the tracks were implemented by using both three- and four-milestoned cells as elements of the tracks. Here we succeeded to use three-milestoned cells only. The important number of motion rules allowed us to assemble such the elements in very efficient structures. Also note the importance of the choice of side 1. As an example, for the cell 4(4), its side 1 is shared with 3(4).

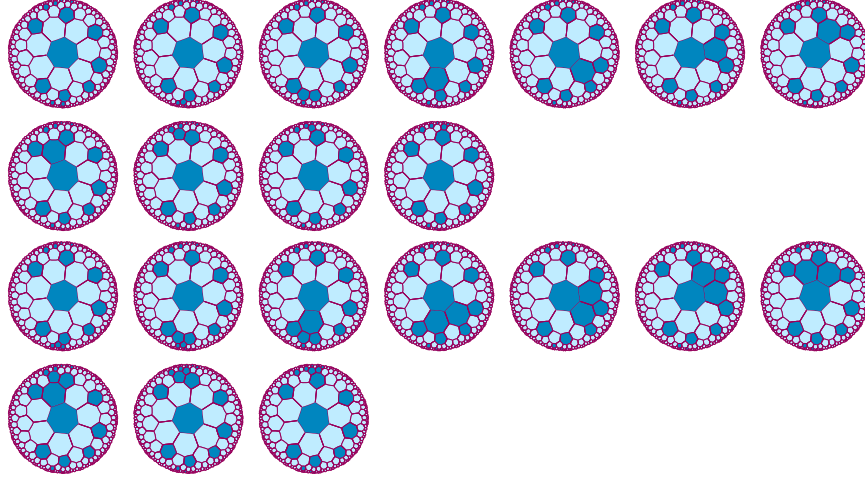
**Table 9** *Execution of the rules 1 up to 66: the double locomotive on the tracks from down to top.*

	11 <sub>4</sub>	10 <sub>4</sub>	3 <sub>4</sub>	1 <sub>4</sub>	1 <sub>5</sub>	1 <sub>6</sub>	1 <sub>7</sub>	1 <sub>1</sub>	3 <sub>1</sub>	7 <sub>1</sub>	6 <sub>1</sub>
1	25	51	47	28	7	4	4	3	4	3	7
2	4	29	51	52	32	4	4	3	4	3	7
3	4	7	29	53	54	36	4	3	4	3	7
4	4	7	7	34	51	56	36	3	4	3	7
5	4	7	7	14	29	48	56	38	4	3	7
6	4	7	7	14	7	25	48	59	41	3	7
7	4	7	7	14	7	4	25	60	61	38	7
8	4	7	7	14	7	4	4	43	48	59	45

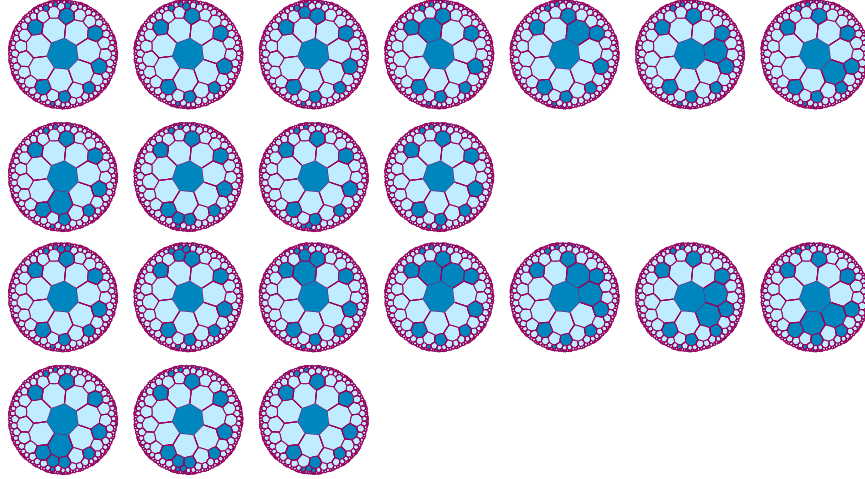
**Table 10** *Execution of the rules 1 up to 66: the double locomotive on the tracks from top to bottom.*

	6 <sub>1</sub>	7 <sub>1</sub>	3 <sub>1</sub>	1 <sub>1</sub>	1 <sub>7</sub>	1 <sub>6</sub>	1 <sub>5</sub>	1 <sub>4</sub>	3 <sub>4</sub>	10 <sub>4</sub>	11 <sub>4</sub>
1	34	51	65	32	7	7	3	7	4	4	7
2	14	29	53	54	16	7	3	7	4	4	7
3	14	7	34	51	47	16	3	7	4	4	7
4	14	7	14	29	51	47	38	7	4	4	7
5	14	7	14	7	29	51	59	45	4	4	7
6	14	7	14	7	7	29	60	66	36	4	7
7	14	7	14	7	7	7	43	51	56	36	7
8	14	7	14	7	7	7	3	29	48	56	16

As mentioned in the introduction, the pictures which constitute Figures 9 and 10 are produced by PostScript programs computed by a computer program. The program applies the rules given in Table 2 and the other tables of rules given in the following sub sections to the cellular automaton. From those calculations, it computes the position of the cell(s) representing the locomotive on the tracks. The same program did the same for the various configurations we shall further investigate.



**Figure 9** *Illustration of the motion from down to top: above, for a simple locomotive, below, for a double locomotive.*



**Figure 10** *Illustration of the motion from top to bottom: above, for a simple locomotive, below, for a double locomotive.*

### 3.2 The rules for the fixed switch, for the fork and for the doubler

We now turn to the study of the fixed switch, of the fork and of the doubler. Table 11 gives new rules which are used for the crossing of those structures together with already used rules. We start our study with the fixed switch

which is a passive structure as noted in Subsection 2.2.

### 3.2.1 The fixed switch

As can already be seen on Figure 3, the structure is mainly constituted by elements of the tracks assembled in a suited way. In particular, the central cell is a three-milestoned cell as in the elements of the tracks. Its neighbourhood is a rotated image of any neighbourhood of a cell of the tracks as studied in Subsection 3.1.

**Table 11** Rules for the crossing of a fixed switch, of a doubler and of a fork.

fixed:	from the left		from the right	
	simple	double	simple	double
67	<u>WWBWWBWW</u>	70 <u>WBBWWBWB</u>	71 <u>WWWBBWBW</u>	72 <u>BWWBBWBW</u>
68	<u>WWBWWBWB</u>			
69	<u>WBBWWBWW</u>			
doubler:	73 <u>BWWBWBWB</u>		fork:	76 <u>WWWBWBWB</u>
	74 <u>WBBBWBWB</u>			77 <u>WBBWBWWW</u>
	75 <u>BWBWWBWB</u>			

Table 13 shows the instructions applied during the crossing of a simple locomotive through the fixed switch from the left and from the right, for the cells of the tracks only. We can notice that the rules involved in the table are those of the motion of the locomotive on the tracks. The information of those tables is completed by that of Table 12 which shows the rules applied at the cell 1(1) which has a view on each track arriving to the central cell. For that latter table, we again used the colours distinguishing the front from the rear in a double locomotive.

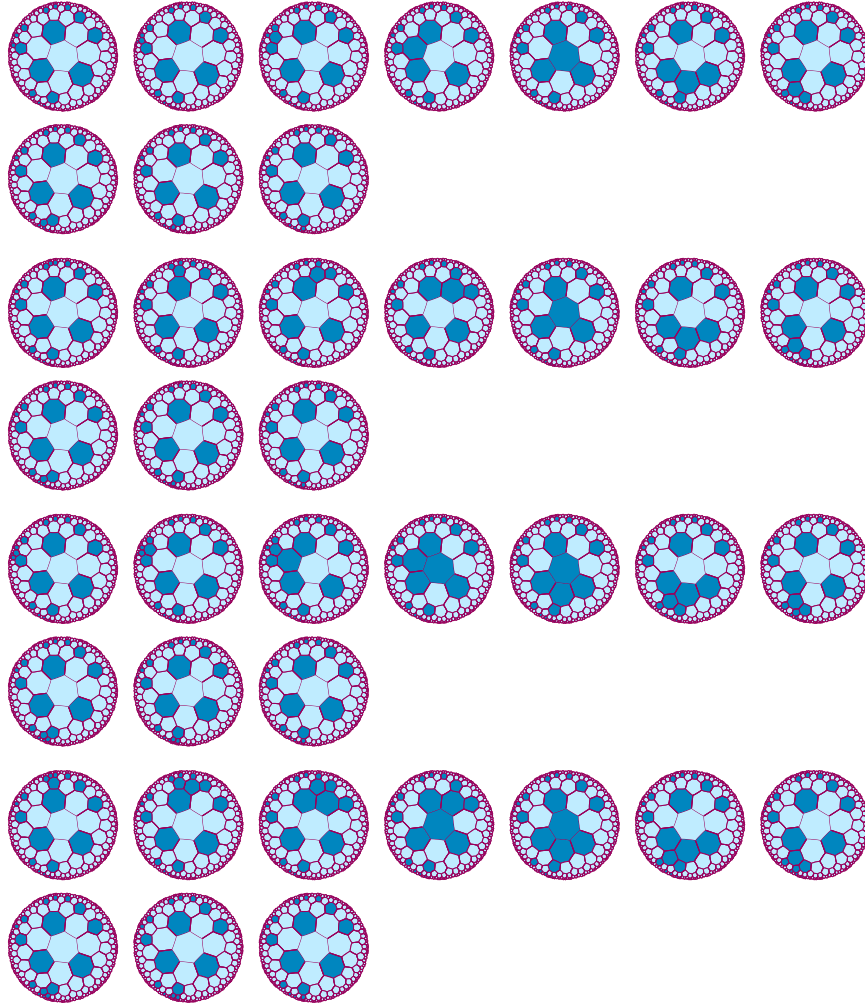
**Table 12** Rules for the cell 1(1) which witnesses the motion on the tracks.

simple, left	double, left	simple, right	double, right
2 <u>BWWWWWWB</u>	2 <u>BWWWWWWB</u>	2 <u>BWWWWWWB</u>	2 <u>BWWWWWWB</u>
44 <u>BWWWWWB</u>	44 <u>BWWWWWB</u>	15 <u>BWBWWWWB</u>	15 <u>BWBWWWWB</u>
23 <u>BWWWWWB</u>	62 <u>BWWWWWB</u>	22 <u>BWBWWWWB</u>	46 <u>BWBWWWWB</u>
2 <u>BWWWWWWB</u>	23 <u>BWWWWWB</u>	19 <u>BWBWWWWB</u>	49 <u>BWBWWWWB</u>
	2 <u>BWWWWWWB</u>	23 <u>BWBWWWWB</u>	50 <u>BWBWWWWB</u>
		2 <u>BWWWWWWB</u>	23 <u>BWBWWWWB</u>
			2 <u>BWWWWWWB</u>

Table 13 shows us the rules applied to the elements of the tracks traversed by the locomotive when it crosses the switch. Table 13 deals with the locomotive, whether it is simple or double. For each side of the switch, the table indicates the group of cells involved by the arrival of the locomotive to the central cell.

We can see that the rules applied to the central cell have already be seen

in Subsection 3.1. The rules for a simple locomotive coming from the left are those of the cell 10(4) in Table 4. For a double locomotive from the left, the rules appear in Tables 6 and 7 for cells whose neighbourhood is 2, 5, 7. For a simple locomotive coming from the right, the front rule used here is in Table 3 with the neighbourhood 2, 5, 7. When a double locomotive comes from the right, Tables 6 and 7 indicate the corresponding rules. Figure 11 illustrates the motion of the locomotive: simple or double, whichever the side from which it arrives to the switch.



**Figure 11** *Illustration of the motion of a locomotive through a fixed switch.*



**Table 13** *Execution of the rules for the fixed switch when a locomotive crosses the switch: upper half, simple locomotive; lower half, double locomotive.*

simple locomotive														
	from the left				from the right				leaving the cell					
	10 <sub>2</sub>	3 <sub>2</sub>	1 <sub>2</sub>	0 <sub>0</sub>	3 <sub>1</sub>	2 <sub>1</sub>	1 <sub>7</sub>	0 <sub>0</sub>	1 <sub>4</sub>	2 <sub>4</sub>	7 <sub>4</sub>	8 <sub>4</sub>	9 <sub>4</sub>	
1	24	38	7	7	40	41	3	7	7	14	7	4	7	
2	29	40	45	7	43	17	38	7	7	14	7	4	7	
3	7	43	24	16	3	25	40	45	7	14	7	4	7	
4	7	3	29	24	3	4	43	24	16	14	7	4	7	
5	7	3	7	29	3	4	3	29	24	63	7	4	7	
6	7	3	7	7	3	4	3	7	29	31	32	4	7	
7	7	3	7	7	3	4	3	7	7	34	24	36	7	
8	7	3	7	7	3	4	3	7	7	14	29	17	16	

double locomotive														
	from the left				from the right				leaving the cell					
	10 <sub>2</sub>	3 <sub>2</sub>	1 <sub>2</sub>	0 <sub>0</sub>	3 <sub>1</sub>	2 <sub>1</sub>	1 <sub>7</sub>	0 <sub>0</sub>	1 <sub>4</sub>	2 <sub>4</sub>	7 <sub>4</sub>	8 <sub>4</sub>	9 <sub>4</sub>	
1	51	59	45	7	60	61	38	7	7	14	7	4	7	
2	29	60	66	16	43	48	59	45	7	14	7	4	7	
3	7	43	51	47	3	25	60	66	16	14	7	4	7	
4	7	3	29	51	3	4	43	51	47	63	7	4	7	
5	7	3	7	29	3	4	3	29	51	65	32	4	7	
6	7	3	7	7	3	4	3	7	29	53	54	36	7	
7	7	3	7	7	3	4	3	7	7	34	51	56	16	
8	7	3	7	7	3	4	3	7	7	14	29	48	47	

Figure 11 illustrates the four motions we have to consider for the fixed switch for which Tables 13 gives the rules used for such motions.

### 3.2.2 The rules for the doubler and for the fork

As clear from Table 11, a few rules only are needed by the doubler and by the fork. As the doubler contains both the fork and the fixed switch, Table 11 displays the three additional rules required by the doubler before the two ones required by the fork as tested in the configuration of the left-hand side picture of Figure 4. Here, we distinguish between the two locomotives created by the cell 4(1) by giving them colours: green for the locomotive which will follow the green path, dark pink for the one which will go along the pink path.

In Table 14, we give the rules used by the doubler when the locomotives cross the cells 4(1), 1(1) and 0(0). Note that when the locomotive enters the cell 4(1), at the next time, two locomotives leave the cell, which is witnessed by the cell, see rule 74 in Table 14. The cell 1(1) also witnesses the duplication

as shown by rules 37, 73, 75 and 50. That last rule witnesses the junction of the two simple locomotives into a double one. Note that the cell 4(1) is applied a sequence of rules which differ from a sequence indicated in Subsection 3.1 by the witness rule: instead of rule 29 as in the motion rules, we have here rule 74 as the cell can see two locomotives created in its neighbours 1 and 3. In the cell 0(0), rule 66 witnesses the junction of the two simple locomotives into a double one. This is why in the sequence of rules 7, 16, 47, 51 and 29 in the crossing of a cell by a double locomotive, see the rules for the cell 1(6) in Table 10, rule 47 is replaced by rule 66 as the occurrence of the second cell, the rear, appears from the side which is opposite to the expected one.

**Table 14** Rules for the cells 4(1), 0(0) and 1(1) which witness the motion of the locomotives in the doubler.

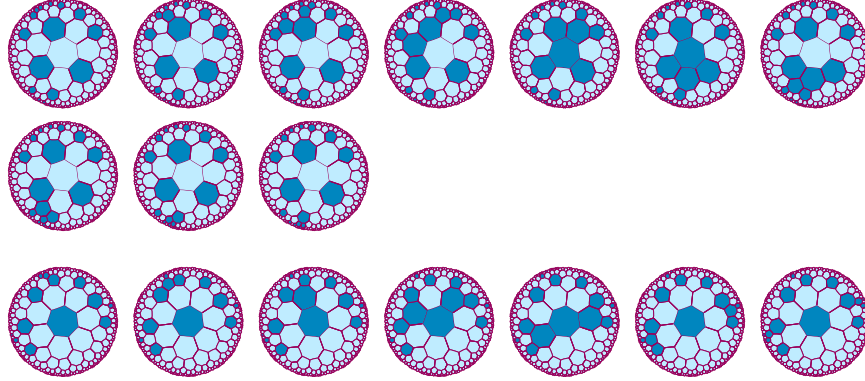
4(1):	0(0):	1(1):
7 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> W	7 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> W	2 <u>B</u> W <u>W</u> W <u>W</u> W <u>B</u>
16 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> <u>B</u>	16 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> <u>B</u>	37 <u>B</u> W <u>W</u> W <u>W</u> W <u>B</u>
24 <u>B</u> W <u>B</u> W <u>B</u> W <u>B</u> W	66 <u>B</u> W <u>B</u> W <u>B</u> W <u>B</u> <u>B</u>	73 <u>B</u> W <u>B</u> W <u>B</u> W <u>B</u>
74 <u>W</u> <u>B</u> <u>B</u> W <u>B</u> W <u>B</u> W	51 <u>B</u> <u>B</u> W <u>B</u> W <u>B</u> W	75 <u>B</u> W <u>B</u> W <u>W</u> W <u>B</u>
7 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> W	29 <u>W</u> <u>B</u> <u>B</u> W <u>B</u> W <u>B</u> W	50 <u>B</u> W <u>W</u> W <u>W</u> W <u>B</u>
	7 <u>W</u> W <u>B</u> W <u>B</u> W <u>B</u> W	2 <u>B</u> W <u>W</u> W <u>W</u> W <u>B</u>

**Table 15** Upper, lower part: execution of the rules for the doubler, the fork respectively, corresponding to the illustrations of Figure 4.

doubler													
	12 <sub>4</sub>	4 <sub>1</sub>	2 <sub>2</sub>	1 <sub>2</sub>	0 <sub>0</sub>	1 <sub>7</sub>	2 <sub>1</sub>	3 <sub>1</sub>	1 <sub>4</sub>	2 <sub>4</sub>	7 <sub>4</sub>	8 <sub>4</sub>	
1	25	24	36	4	7	3	7	41	7	14	7	4	
2	4	74	17	36	7	3	16	17	7	14	7	4	
3	4	7	25	17	16	38	24	25	7	14	7	4	
4	4	7	4	25	66	60	29	4	16	14	7	4	
5	4	7	4	25	51	43	7	4	47	63	7	4	
6	4	7	4	4	29	3	7	4	51	65	32	4	
7	4	7	4	4	7	3	7	4	29	53	54	36	
8	4	7	4	4	7	3	7	4	7	34	51	56	

fork												
	10 <sub>1</sub>	4 <sub>1</sub>	1 <sub>1</sub>	1 <sub>2</sub>	1 <sub>3</sub>	3 <sub>3</sub>	7 <sub>3</sub>	1 <sub>7</sub>	1 <sub>6</sub>	4 <sub>6</sub>	5 <sub>7</sub>	
1	29	17	16	4	7	7	4	4	4	4	7	
2	7	25	24	36	7	7	4	41	4	4	7	
3	7	4	74	17	16	7	4	17	36	4	7	
4	7	4	7	25	24	16	4	25	17	36	7	
5	7	4	7	4	29	24	36	4	25	17	16	



**Figure 12** *Illustration of the crossing of the doubler, above, and of the fork, below, by the locomotive.*

### 3.3 The rules for the selector

The selector is the last structure we need to implement roundabouts. The new rules needed by the structure are given in Table 16 while the execution of the rules used by the crossing of a locomotive are given in Table 17: the left-, right-hand side sub-table gives the rules used by a simple, double locomotive respectively.

**Table 16** *Rules for the locomotive through the selector.*

simple locomotive							
78	<u>WBWBBWBW</u>	85	<u>BBWBWWBB</u>	92	<u>BBWBBWBW</u>	99	<u>WBBWWBBW</u>
79	<u>WBBWBWBW</u>	86	<u>WBWBBWBW</u>	93	<u>BBWWBBBW</u>	100	<u>BWBBWBWB</u>
80	<u>BWBWBWBW</u>	87	<u>BWWWBWBW</u>	94	<u>WBBBBBBW</u>	101	<u>BWBBWBWB</u>
81	<u>BBWBWBWB</u>	88	<u>BBWBWBWB</u>	95	<u>BBWBWBWB</u>	102	<u>BWBWBWBW</u>
82	<u>BBWBWBWB</u>	89	<u>BBWBWBWB</u>	96	<u>WBBBBBBW</u>	103	<u>BWBBWBWB</u>
83	<u>WBBWBWBW</u>	90	<u>BWBBWBWB</u>	97	<u>BWWWBWBW</u>	104	<u>WBBBBBBW</u>
84	<u>BBWBWBWB</u>	91	<u>BBWBWBWB</u>	98	<u>WBBWBWBW</u>		
double locomotive							
105	<u>BBWBWBWB</u>	109	<u>BBWBWBWB</u>	113	<u>BBWBWBWB</u>	117	<u>WBBWBWBW</u>
106	<u>BBWBWBWB</u>	110	<u>BBWBWBWB</u>	114	<u>WBBBBBBW</u>	118	<u>WBWBWBWB</u>
107	<u>BBWBWBWB</u>	111	<u>BWBWBWBW</u>	115	<u>BWBWBWBW</u>	119	<u>BBWBWBWB</u>
108	<u>BBWBWBWB</u>	112	<u>BBWBWBWB</u>	116	<u>BWBWBWBW</u>	120	<u>BWBWBWBW</u>

In both sub-tables of Table 17, we can see that the track leading the locomotive to the selector make use of motion rules examined in Sub-section 3.1, the cell 1(6) excepted. That cell, which constitutes the entrance to the selector has a specific neighbourhood involving five milestones.

**Table 17** Execution of the rules for a locomotive passing through the selector.

simple locomotive											
	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	1 <sub>1</sub>	2 <sub>1</sub>	7 <sub>1</sub>	1 <sub>7</sub>	1 <sub>5</sub>	
1	29	17	36	79	78	7	4	7	57	64	
2	7	25	17	83	78	7	4	7	57	64	
3	7	4	25	88	86	7	4	7	90	87	
4	7	4	4	94	92	16	4	7	95	93	
5	7	4	4	99	96	24	36	7	101	98	
6	7	4	4	79	78	29	17	16	103	64	
7	7	4	4	79	78	7	25	24	57	64	

double locomotive											
	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	1 <sub>4</sub>	2 <sub>5</sub>	5 <sub>5</sub>	12 <sub>4</sub>	1 <sub>7</sub>	1 <sub>5</sub>
1	29	48	56	83	78	4	7	7	4	57	64
2	7	25	48	107	86	4	7	7	4	90	87
3	7	4	25	112	110	36	7	7	4	113	111
4	7	4	4	118	114	17	16	7	4	117	116
5	7	4	4	79	78	25	24	16	4	57	120
6	7	4	4	79	78	4	29	24	36	57	64
7	7	4	4	79	78	4	7	29	17	57	64

**Table 18** Rules for the cells 1(7) and 1(5), 1(6) and 0(0) which witness the motion of the locomotives in the doubler.

1(7)				1(5)			
simple		double		simple		double	
57	<u>BWWBBWWB</u>	57	<u>BWWBBWWB</u>	64	<u>BWWWWBBWB</u>	64	<u>BWWWWBBWB</u>
90	<u>BWB<sup>B</sup>BBWWB</u>	90	<u>BWB<sup>B</sup>BBWWB</u>	87	<u>BWWWWBB<sup>B</sup>B</u>	87	<u>BWWWWBB<sup>B</sup>B</u>
95	<u>B<sup>B</sup>WB<sup>B</sup>BBWWB</u>	113	<u>B<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup></u>	93	<u>B<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup></u>	111	<u>B<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup></u>
101	<u>BWWBBWW<sup>B</sup>B</u>	117	<u>W<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup>B</u>	98	<u>W<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup>B</u>	116	<u>B<sup>B</sup>WB<sup>B</sup>BBWW<sup>B</sup>B</u>
103	<u>BWWBBWB<sup>B</sup>B</u>	57	<u>BWWBBWWWB</u>	64	<u>BWWWWBBWB</u>	120	<u>BWW<sup>B</sup>WB<sup>B</sup>BBWB</u>
57	<u>BWWBBWWWB</u>					64	<u>BWWWWBBWB</u>

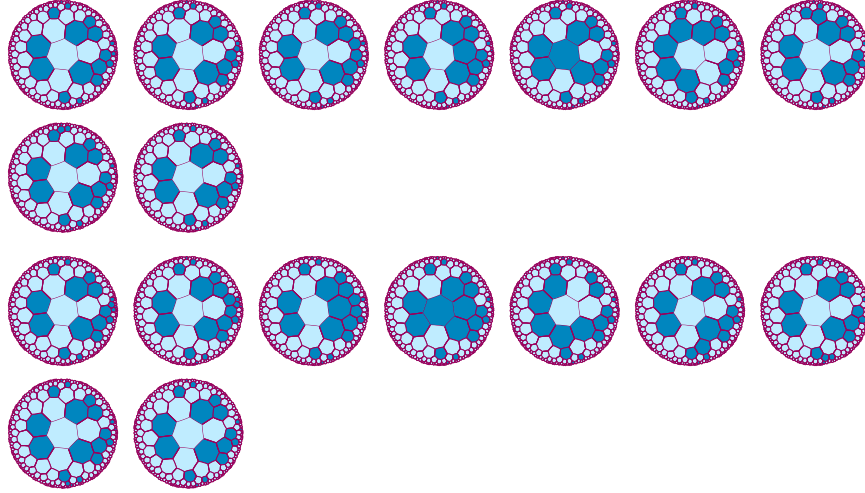
1(6)				0(0)			
simple		double		simple		double	
79	<u>WBBWBBBWW</u>	79	<u>WBBWBBBWW</u>	78	<u>WBWBBWBWW</u>	78	<u>WBWBBWBWW</u>
83	<u>WBBWBBB<sup>B</sup>B</u>	83	<u>WBBWBBB<sup>B</sup>B</u>	86	<u>WBWBBWB<sup>B</sup>B</u>	86	<u>WBWBBWB<sup>B</sup>B</u>
88	<u>B<sup>B</sup>WB<sup>B</sup>BBBWW</u>	107	<u>B<sup>B</sup>WB<sup>B</sup>BBB<sup>B</sup>B</u>	92	<u>B<sup>B</sup>WB<sup>B</sup>BBBWW</u>	110	<u>B<sup>B</sup>WB<sup>B</sup>BBBWW</u>
94	<u>WBBB<sup>B</sup>BBBWW</u>	112	<u>B<sup>B</sup>B<sup>B</sup>BBBWW</u>	96	<u>W<sup>B</sup>B<sup>B</sup>B<sup>B</sup>WW</u>	114	<u>W<sup>B</sup>B<sup>B</sup>B<sup>B</sup>WW</u>
99	<u>WBBW<sup>B</sup>BBWW</u>	118	<u>W<sup>B</sup>W<sup>B</sup>BBBWW</u>	78	<u>WBWBBWBWW</u>		<u>WBWBBWBWW</u>
79	<u>WBBWBBBWW</u>	79	<u>WBBWBBBWW</u>				

Among them, the cells 1(7) and 1(5) which constitute the sensors of the selector: they detect whether a simple or a double locomotive arrived at the

cell 0(0). Table 18 shows the rules applied at the cells 1(7), 1(5), 1(6) and 0(0). For each cell, the table gives the rules when a simple locomotive arrives and then, when a double one arrives. Note that **w** indicates that the cell 1(7) or 1(5) became white for one time in order to cancel the locomotive prepared for the corresponding path.

Figure 13 illustrates the motion of the locomotive in the selector, whether it is simple or double.

At this point, it is important to point at the fact that for the cell 1(6), side 1 is not shared with 0(0) but with a milestone of 1(6), namely 2(7). In the same way, for 0(0), side 1 is not shared with a cell of the tracks, it is also shared with a milestone, with the cell 1(7). The reason of these choices lies in the fact that the idle neighbourhood of 1(6) coincide with the rotated form of a neighbourhood of 0(0) in the selector: this can be seen with rules 79 and 86 whose neighbourhood are rotated forms of each other and which, consequently, are rotationally incompatible.



**Figure 13** *Illustration of the crossing of the selector: above, by a simple locomotive, below, by a double one.*

### 3.4 The rules for the controller

Let us now consider the rules for the controller of the active switches. The rules are displayed by Table 19. As mentioned in the table itself, the two columns in the left-hand side deal with the passage of the locomotive while the last column deals with the change of colour of the controller. We remind the reader that the colour of the controller is the colour of the cell 1(3) in Figure 6. Table 20 and Figure 19 illustrate the crossing of a black controller by the locomotive. All cells of the track obey the rules we have considered for the tracks for three-

milestoned rules. As an example, the cell 1(4) is applied the same rules as the cell 1(6) in Tables 1 and 4. We shall look closely at the cell 1(3), the control cell of the structure and at the central cell 0(0).

**Table 19** Rules for the control: passage of the locomotive and signal for changing the selected track.

passage of the locomotive		signal	
black	white	B → W	W → B
121 <u>WWWWWWBBW</u>	125 <u>WWWWWWBWW</u>	129 <u>BWWBBBWWW</u>	
122 <u>WBWBWWWWW</u>	126 <u>WWBBBWWWW</u>		W → B
123 <u>WBWBWWWWW</u>	127 <u>WBWBWBWWW</u>		
124 <u>WBBWBWWWW</u>	128 <u>WWWWBWWWW</u>	130 <u>WBWBWBWBW</u>	
		131 <u>WWBBBWWWB</u>	
		132 <u>BWBWBWBWW</u>	

**Table 20** Execution of the rules used during the traversal of a black controller by the locomotive.

	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	1 <sub>4</sub>	2 <sub>5</sub>	5 <sub>5</sub>	12 <sub>4</sub>	1 <sub>3</sub>
1	29	17	63	4	4	4	7	7	4	57
2	7	25	31	45	4	4	7	7	4	57
3	7	4	34	17	36	4	7	7	4	57
4	7	4	14	25	17	36	7	7	4	95
5	7	4	14	4	25	17	16	7	4	101
6	7	4	14	4	4	25	24	16	4	57
7	7	4	14	4	4	4	29	24	36	57

**Table 21** Rules for the cells 0(0) and 1(3) of the controller. Rules for the passage of the locomotive and for the signal, whatever the colour of the controller.

	passage		signal	
0(0):	black	white	black	white
	4 <u>WBWBWBWBW</u>	77 <u>WBWBWBWBW</u>	4 <u>WBWBWBWBW</u>	77 <u>WBWBWBWBW</u>
	36 <u>WBWBWBWBW</u>	104 <u>WBWBWBWBW</u>	77 <u>WBWBWBWBW</u>	4 <u>WBWBWBWBW</u>
	17 <u>BWBWBWBWBW</u>	77 <u>WBWBWBWBW</u>		
	25 <u>BWBWBWBWBW</u>			
	4 <u>WBWBWBWBW</u>			
1(3):	black	white	black	white
	57 <u>BWBWBWBWB</u>	126 <u>WBWBWBWBW</u>	57 <u>BWBWBWBWB</u>	126 <u>WBWBWBWBW</u>
	95 <u>BWBWBWBWB</u>		129 <u>BWBWBWBWB</u>	131 <u>WBWBWBWBW</u>
	101 <u>BWBWBWBWB</u>		126 <u>WBWBWBWBW</u>	57 <u>BWBWBWBWB</u>

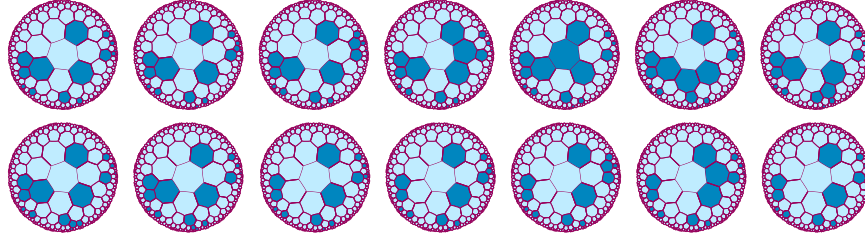
In Table 21, as in previous tables, black and white cells have different meaning with respect to the simulation. For the convenience of the reader, we indicate that **B** marks the locomotive, **B** marks the signal, **B** shows us that the controller

is black which allows the passage of the locomotive while **w** shows us that it is white which forbids the passage of the locomotive. The way the rules are working should now be clear without further comments.

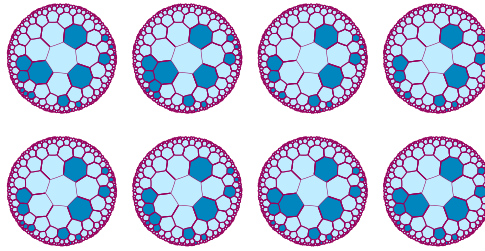
Table 22 indicates the rules which are applied when the locomotive arrives at a white controller and those which are applied when the signal for changing its colour arrives at the controller.

**Table 22** *Execution of the rules when the locomotive arrives to a white controller and when the signal for changing the colour arrives.*

locomotive, 1(3) white						signal for changing the colour									
	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	W → B					B → W				
1	29	17	63	4	77	12 <sub>3</sub>	4 <sub>3</sub>	1 <sub>3</sub>	1 <sub>2</sub>		12 <sub>3</sub>	4 <sub>3</sub>	1 <sub>3</sub>	1 <sub>2</sub>	
2	7	25	31	45	77	1	25	132	131	125	1	25	92	129	121
3	7	4	34	17	104	2	4	78	57	121	2	4	127	126	125



**Figure 14** *Illustration of the crossing of the controller by the locomotive: above and first figure of the second row, when it is black, below after the second figure, when it is white.*



**Figure 15** *Illustration of the arrival of the signal to the controller. Above: from black to white; below: from white to black.*

Figure 14 illustrates the crossing of controller by the locomotive in both

cases, according to the colour of 1(3). Figure 15 illustrates the arrival of the signal for changing the colour of the controller.

### 3.5 The rules for the sensor

In this last subsection of Section 3, we examine the rules which manage the working of the sensor, the specific control structure of the passive memory switch. Sub-section 2.3 in Section 2 explained the working of the structure, pointing at the differences between the controller and the sensor illustrated by Figures 6 and 7.

Table 23 illustrates the few rules which have to be appended to the already examined 132 ones in order to make the structure working as expected.

**Table 23** *Rules for the sensor of the passive memory switch.*

		passage		signal	
		white		B → W	
133	<u>WBWBWBWWW</u>	139	<u>WBWBWBWBW</u>		
134	<u>WWWBWBWWW</u>	140	<u>WBWBWBWBW</u>		
135	<u>WBWBWBWBW</u>		black	143	<u>BWWWBWBW</u>
136	<u>BWBWBWBWW</u>	141	<u>WBWBWBWBW</u>	144	<u>BBBBWWWB</u>
137	<u>BWWWBWBW</u>	142	<u>WBWBWBWBW</u>		
138	<u>WBWBWBWBW</u>				

As can be seen in the comparison of Figures 6 and 7, many rules used for the controller are also used for the sensor. As an example, as long as the sensor is white, the rules executed in the cells of the tracks when the locomotive passes are the same as those used in the same action when the controller is black, see Tables 20 and 24.

**Table 24** *Execution of the rules when the sensor is white and then a locomotive passes.*

	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	1 <sub>4</sub>	2 <sub>5</sub>	5 <sub>5</sub>	12 <sub>4</sub>	1 <sub>1</sub>
1	29	17	63	4	133	4	7	7	4	134
2	7	25	31	45	133	4	7	7	4	134
3	7	4	34	17	135	4	7	7	4	134
4	7	4	14	25	136	36	7	7	4	131
5	7	4	14	4	138	17	16	7	4	58
6	7	4	14	4	139	25	24	16	4	58
7	7	4	14	4	139	4	29	24	36	58

However, as shown by Figures 6 and 7, in the sensor, 1(3) is a milestone which, accordingly, is permanently black. The sensor cell is 1(1) which is either white or black, depending on the current function of the sensor. We remind



that the locomotive passes only if the sensor is white: this means that the passing locomotive becomes a signal which in the sensor provokes the change of the cell 1(1) from white to black and which will reach the other sensor as a signal which will provoke the change of the other cell 1(1) from black to white. When 1(1) is black and the locomotive arrives, it is stopped as the current locomotive passed through the selected track arriving to the passive memory switch: nothing has to be changed so that the duplicated locomotive has not be changed into a signal to the other sensor.

Table 25 shows the rules used for the cells 0(0) and 1(1). We use the same conventions of colours as for the controller: as the global meaning of **B** and of **W** are the same, we keep the marks **B** and **W** for the cell 1(1) which is the sensor cell of the structure.

**Table 25** Rules for the cells 0(0) and 1(3) of the controller. Rules for the passage of the locomotive and for the signal, whatever the colour of the controller.

	passage		signal
0(0):	white	black	black
	133 <u>WBWBW<b>W</b>W</u>	139 <u>WBWBW<b>B</b>W</u>	139 <u>WBWBW<b>B</b>W</u>
	135 <u>WBWB<b>B</b>W<b>B</b></u>	141 <u>WBWB<b>B</b>W</u>	133 <u>WBWBW<b>B</b>W</u>
	136 <u><b>B</b>WBWBW<b>W</b></u>	139 <u>WBWBW<b>B</b>W</u>	
	138 <u>WB<b>B</b>WBW</u>		
	139 <u>WBWBW<b>B</b>W</u>		
1(1):	white	black	black
	134 <u>W<b>W</b>WBBW</u>	58 <u><b>B</b>W<b>W</b>BBW</u>	58 <u>BW<b>W</b>BBW</u>
	131 <u>W<b>W</b>WBBW</u>		143 <u>BW<b>W</b>BBW</u>
	58 <u>BW<b>W</b>BBW</u>		134 <u>W<b>W</b>WBBW</u>

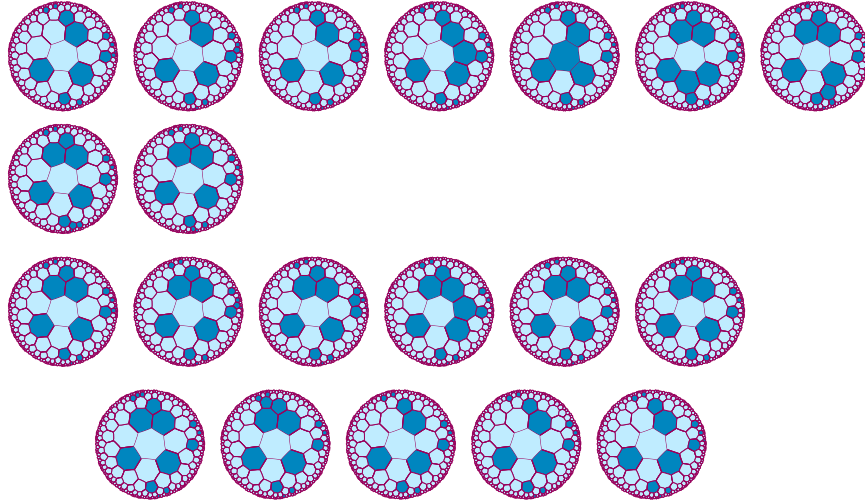
**Table 26** Execution of the rules for the black sensor, for the locomotive and for the signal.

	locomotive					signal			
	9 <sub>6</sub>	10 <sub>6</sub>	4 <sub>6</sub>	1 <sub>6</sub>	0 <sub>0</sub>	9 <sub>1</sub>	3 <sub>1</sub>	1 <sub>1</sub>	2 <sub>1</sub>
1	29	<b>17</b>	<b>63</b>	4	139	1	29	<b>92</b>	<b>143</b> 144
2	7	25	<b>31</b>	<b>45</b>	139	2	7	127	134 15
3	7	4	34	<b>17</b>	141	3	7	127	134 15
4	7	4	14	4	139				

Table 25 shows us several features. First, due to the choice of side 1 shared with 1(3), the neighbourhood of 0(0) is now 1, 3, 5, see rule 133 in Tables 23 and 25. However, after the applications of rules 135 and 136 working as front and cell rules, the next rule is 138 as the passage of the locomotive in 0(0) triggered the change of 1(1) from **W** to **B**, see rule 138 in Table 25 followed by rule 139, the conservative rule of 0(0) when the controller is black. This can be seen in the column devoted for 0(0) to the passage of the locomotive in a black controller.

A second feature deals with the cell 1(1): its neighbourhood is 4, 5. The rule is conservative when the sensor is black, see rule 58, it is not when it is white : see rule 131. Cells with only two milestones and contiguous ones were already met in the motion on the tracks as rules 18 and 21 or rules 121 and 126 for the controller. In all those latter rules which are conservative the current state of the cell is white. This again point at the importance of the relaxation of the rotation invariance hypothesis.

The third feature is another fact about the rules for 1(1) in Table 25: the change of coloured is not triggered in the same way. From white to black, the change is triggered by the locomotive, see rule 131 where the black cell of the locomotive is marked with **B**. From black to white, it is triggered by the signal, see rule 143 where the locomotive-signal is marked **B**.



**Figure 16** *Illustration of the working of the sensor. First row: passage of the locomotive; second row: the locomotive is stopped; third row: the signal changes a black sensor to a white one.*

We completed the examination of the rules for the sensor. Accordingly, Theorem 1 is proved.  $\square$

## Conclusion

As mentioned in the introduction, Theorem 1 is the best result for tessellations involving regular convex polygon with the angle  $\frac{2\pi}{3}$ : the tessellation  $\{7, 3\}$  is the tessellation  $\{p, 3\}$  where  $p$  has the smallest value as possible for the hyperbolic plane. With this model, the implementation in the heptagrid seems to be impossible under the rotation invariance assumption. There are two rules with no black neighbour and also two ones with exactly one black neighbour. The

neighbourhoods are:

$$w^7 \text{ and } Bw^6.$$

There are also two rules with no white neighbour and two ones with one white neighbour exactly: they are obtained by **contraposition** from the just mentioned ones, *i.e.* by exchanging  $B$  and  $w$ . There are six rules with either two black neighbours exactly, which are

$$B^2w^5, BwBw^4 \text{ and } Bw^2Bw^3.$$

The neighbours with two white neighbours exactly are obtained by contraposition. There are ten rules three black neighbours exactly:

$$B^3w^4, B^2wBw^3, B^2w^2Bw^2, B^2w^3Bw \text{ and } BwBwBw^2.$$

By contraposition we obtain the neighbourhoods with three white neighbours exactly. Accordingly, there at most forty rotation independent rules. There are already big problems to define the tracks as our solution with three milestones produces a single rotation invariant rule and, as noted in Subsection 3.1, several rules are rotationally incompatible. With four-milestoned rules, we have a problem with symmetric patterns which cannot be used for one way motions. For the same reason of symmetry, we cannot use a pattern with five milestones as the two white neighbours for entry and exit could not be distinguished.

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